# Grounding Mathematics in Formal Ideas : An Empirical Synthesis

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## INTRODUCTION

**How do abstract mathematical concepts gain meaning?**

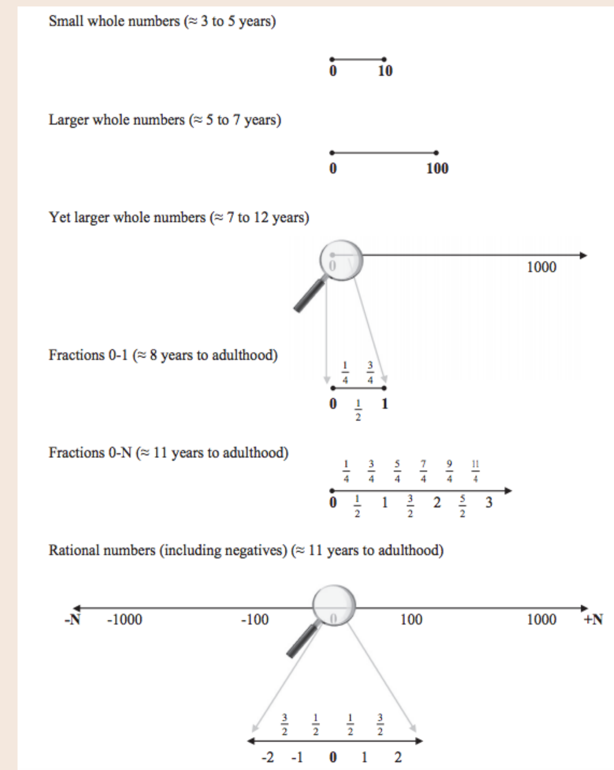
Contemporary cognitive science research suggests that numbers and the arithmetic operations performed on them are grounded in innate perceptual and motor systems. In addition to these two explanations, we propose adding a third mechanism – ***formal anchoring***. This hypothesis emphasizes how **abstract numbers and arithmetic operations themselves can themselves bootstrap an understanding of higher-level concepts and procedures.** In this way, the abstract becomes concrete.

PERCEPTION | ACTION | FORMAL IDEAS

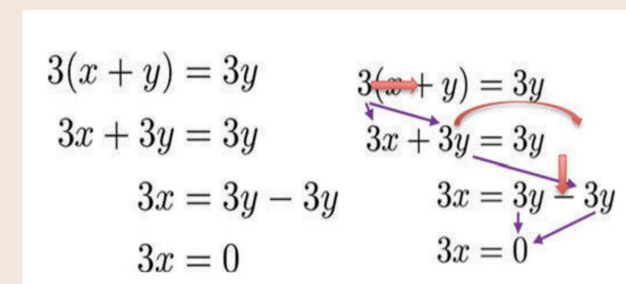
## DOMINANT THEORIES

Dominant accounts of numerical and arithmetic development emphasize the recruitment, repurposing, and refinement of perceptuo-motor and visuospatial systems (Blair, Tsang, & Schwartz, 2013; Landy, Allen, & Zednik, 2014; Siegler, 2016).

### INTEGRATED THEORY

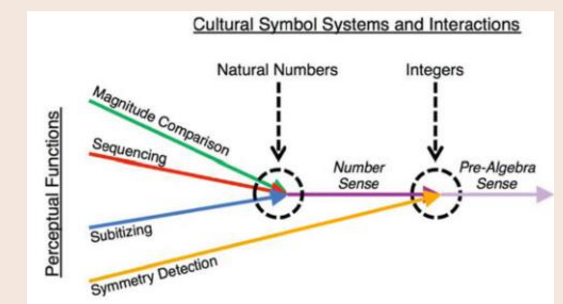



### PERCEPTUAL MANIPULATIONS




"complex visual and auditory processes such as affordance learning, **perceptual pattern-matching and perceptual grouping of notational structures produce simplified representations of the mathematical problem, simplifying the task faced by the rest of the symbolic reasoning system**. Perceptual processes exploit the typically well-designed features of physical notations to automatically reduce and simplify difficult, routine formal chores, and so are themselves constitutively involved in the capacity for symbolic reasoning (Landy, Allen, & Zednick, 2014)."

### BUNDLING HYPOTHESIS




"**integrating perceptuo-motor capacities** ensures that children learn that 5 big, 5 total, and 5th have a tight relation and that a change within one type of meaning...entails changes within the other types (6 is more than 5)... it has empirical support. **Children who went through instruction that focused on integrating order, counting, and magnitude did better in math the following year than students whose curricula emphasized each sense of number separately** (Schwartz, Blair, & Tsang, 2012)."

## GAPS & GOALS

Current theories of numerical and arithmetic cognition are specific to particular tasks and number systems. To date, it is unclear how well existing theories generalize across task types and number systems. We seek a **theory of mathematical cognition** robust enough to :

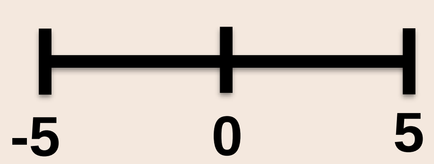
1. **Explain mathematical cognition phenomena across multiple ecologically valid tasks and numbers systems of varying abstraction**
2. **Generate novel instructional methods for improving mathematics education**
3. **Extend mathematical cognition research to higher-level topics like complex numbers, algebra, and calculus**

To construct such a theory, we **reviewed numerical cognition literature** discussing how natural numbers, integers, and rational numbers are understood by people of differing ages. In addition, **we supplement the review with our own study examining the cognitive bases of irrational numbers** (Patel & Varma, *in press*).

**MAGNITUDE COMPARISON**

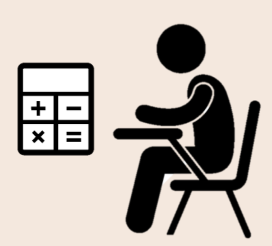

***Compare the lesser or smaller number in a pair quickly and accurately.***

**NUMBER LINE ESTIMATION**

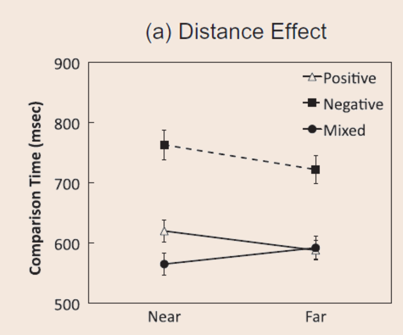



***Mark on a blank number line where various numbers are located quickly and accurately.***

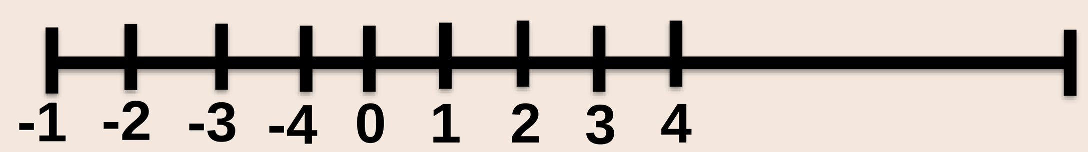
**ARITHMETIC COMPUTATION**

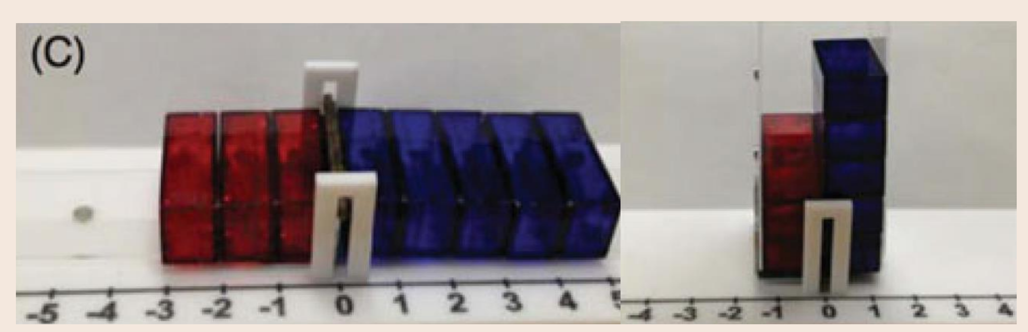


***Solve arithmetic problems accurately, quickly, and efficiently.***




Over time, people shift from using rules to judge the magnitude of integer pairs like (-4, 9) to referencing the anchor zero (Varma & Schwartz, 2011).




Instruction with a number line manipulative that emphasizes the role of zero improves arithmetic learning (Tsang et al, 2015).

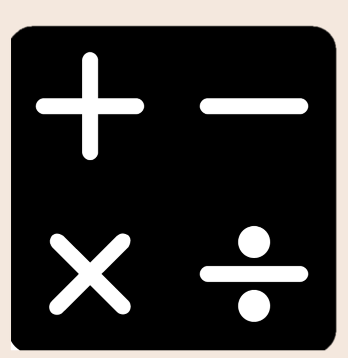
| | Naturals | | Integers | Rationals | | Irrationals |
|---|---|---|---|---|---|---|
| | one-digit | multi-digit | | fractions | decimals | |
| Magnitude Comparison | none | multiples of 10 | 0 | unit fractions, decimals | naturals | naturals[a] |
| Number Line Estimation | 0, 5, 10 | quartiles thirds | 0 | unit fractions, decimals | quartiles tenths | perfect squares[b] |
| Arithmetic Computation | 10 | multiples of 10 | 0 | ? | ? | perfect squares[c] |

[a] Supported by the equivalence strategy on the MC task.
[b] Supported the single and double landmark strategies on the NLE task.
[c] Supported by the perfect squares factorization strategy on the irrational knowledge test.

***Table 1***. Across three major numerical tasks and four number systems, people anchor cognitive processes on more concrete numbers. These *formal anchors* are content-dependent (Patel & Varma, *in press*).

## ARITHMETIC OPERATIONS

***People use addition to understand subtraction:***
***36 – 19 = ? is often transformed to 19 + ? = 36***



(Campbell, 2008)

(Mauro, LeFevre, & Morris, 2003)

***People use multiplication to understand division:***
***56 / 7 = ? is often transformed to 7 * ? = 56***

## FORMAL ANCHORING HYPOTHESIS

1. **Formal anchoring is the task-specific recruitment of more concrete representations.**
2. **It is used to alleviate computational and working memory demands.**
3. **The benefits of formal anchoring are task-specific. It is sometimes necessary for successful performance. In some cases, however, is merely improves how quickly the goal state is reached.**
4. **To explain why particular concrete mathematical concepts are appropriate on some tasks but not others, it is useful to focus on the alignment between what a task demands and what a representation provides. An appropriate representation simplifies a task without altering its underlying nature.**

## FUTURE RESEARCH

1. **Which cognitive and environmental factors promote the discovery and use of formal anchors ?**
2. **Is the use of formal anchors always more efficient ?**
3. **What are the neural bases of formal anchoring ?**
4. **How well can formal anchoring explain mathematical cognition specific to more complex domains like algebra and calculus?**

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